Using Uncertainty Matrices to Analyse Measurements of Multiple S-parameters

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Overview

- Introduction
- Concept
- Example
- Uncertainty Ellipses
- Discussion
- Conclusion
Introduction

- Microwave measurements often contain interactions between multiple measurement parameters.

- For example, for a mismatch line (Beatty line), as input reflection goes up, forward transmission goes down.
Introduction (contd)

In this talk:

- Use uncertainty matrices to represent the interactions between measurement parameters
- Multiple (>1) complex-valued S-parameters

Why uncertainty matrices?

- Because these are useful when propagating uncertainties from one set of parameters to other parameters
Introduction (contd)

- For simplicity (here):
  - Concentrate on ‘just’ a 2-port device
    - Later on, we will assume symmetry ($S_{11} = S_{22}$)
    - And we will assume reciprocity ($S_{21} = S_{12}$)
  - The approach generalizes to $n$-port devices that are non-symmetric and non-reciprocal
**Concept**

**One-port device**

One complex-valued S-parameter, \( S_{11} \equiv (S_{11R}, S_{11I}) \)

Uncertainty matrix:

\[
\begin{pmatrix}
    u^2(S_{11R}) & u(S_{11R}, S_{11I}) \\
    u(S_{11I}, S_{11R}) & u^2(S_{11I})
\end{pmatrix}
\]
Concept (contd)

One-port device

The (2 x 2) uncertainty matrix shows the uncertainty in each component of the complex-valued S-parameter

i.e. $S_{11_R}$ and $S_{11_I}$

\[
\begin{pmatrix}
  u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\
  u(S_{11_I}, S_{11_R}) & u^2(S_{11_I})
\end{pmatrix}
\]
Concept (contd)

One-port device

The (2 x 2) uncertainty matrix also shows the interaction between each component of the complex-valued S-parameter.

i.e. between $S_{11R}$ and $S_{11I}$

$$\begin{pmatrix}
  u^2(S_{11R}) & u(S_{11R}, S_{11I}) \\
  u(S_{11I}, S_{11R}) & u^2(S_{11I})
\end{pmatrix}$$
Two-port device

Four complex-valued S-parameters, $S_{11}$, $S_{21}$, $S_{12}$, $S_{22}$

Four (2 x 2) uncertainty matrices: one for each S-parameter

\[
\begin{pmatrix}
    u^2(S_{11R}) & u(S_{11R}, S_{11I}) \\
    u(S_{11I}, S_{11R}) & u^2(S_{11I})
\end{pmatrix}
\begin{pmatrix}
    u^2(S_{21R}) & u(S_{21R}, S_{21I}) \\
    u(S_{21I}, S_{21R}) & u^2(S_{21I})
\end{pmatrix}
\begin{pmatrix}
    u^2(S_{12R}) & u(S_{12R}, S_{12I}) \\
    u(S_{12I}, S_{12R}) & u^2(S_{12I})
\end{pmatrix}
\begin{pmatrix}
    u^2(S_{22R}) & u(S_{22R}, S_{22I}) \\
    u(S_{22I}, S_{22R}) & u^2(S_{22I})
\end{pmatrix}
\]
Concept (contd)

Two-port device

Each (2 x 2) uncertainty matrix shows the **uncertainty in, and interaction between**, each component in each complex-valued S-parameter, $S_{11}$, $S_{21}$, $S_{12}$, $S_{22}$

\[
\begin{pmatrix}
  u^2(S_{11R}) & u(S_{11R},S_{11I}) \\
  u(S_{11I},S_{11R}) & u^2(S_{11I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{21R}) & u(S_{21R},S_{21I}) \\
  u(S_{21I},S_{21R}) & u^2(S_{21I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{12R}) & u(S_{12R},S_{12I}) \\
  u(S_{12I},S_{12R}) & u^2(S_{12I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{22R}) & u(S_{22R},S_{22I}) \\
  u(S_{22I},S_{22R}) & u^2(S_{22I})
\end{pmatrix}
\]
Concept (contd)

Two-port device

Each (2 x 2) uncertainty matrix shows the **uncertainty in, and interaction between**, each component in each complex-valued S-parameter, \( S_{11}, S_{21}, S_{12}, S_{22} \)

\[
\begin{pmatrix}
  u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\
  u(S_{11_I}, S_{11_R}) & u^2(S_{11_I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{21_R}) & u(S_{21_R}, S_{21_I}) \\
  u(S_{21_I}, S_{21_R}) & u^2(S_{21_I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{12_R}) & u(S_{12_R}, S_{12_I}) \\
  u(S_{12_I}, S_{12_R}) & u^2(S_{12_I})
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u^2(S_{22_R}) & u(S_{22_R}, S_{22_I}) \\
  u(S_{22_I}, S_{22_R}) & u^2(S_{22_I})
\end{pmatrix}
\]
Concept (contd)

Two-port device

But these (2 x 2) uncertainty matrices do not show the interaction between a component of one S-parameter and a component of a different S-parameter.

For example, for the interaction between $S_{11R}$ and $S_{21R}$, we need the uncertainty matrix:

$$
\begin{pmatrix}
    u^2(S_{11R}) & u(S_{11R}, S_{21R}) \\
    u(S_{21R}, S_{11R}) & u^2(S_{21R})
\end{pmatrix}
$$
Concept (contd)

Two-port device

Or, for the interaction between $S_{11R}$ and $S_{21I}$,

we need the uncertainty matrix:

$$
\begin{pmatrix}
    u^2(S_{11R}) & u(S_{11R}, S_{21I}) \\
    u(S_{21I}, S_{11R}) & u^2(S_{21I})
\end{pmatrix}
$$
Concept (contd)

Two-port device

To represent the interaction between:

- both components (real and imaginary)
- of all four S-parameters ($S_{11}$, $S_{21}$, $S_{12}$, $S_{22}$)

We need a single ([4x2] x [4x2]) matrix

≡ (8 x 8) matrix
Concept (contd)

Two-port device

Four complex-valued S-parameters, $S_{11}$, $S_{21}$, $S_{12}$, $S_{22}$

Full uncertainty matrix: one (8 x 8) matrix

<table>
<thead>
<tr>
<th></th>
<th>$S_{11}$</th>
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Concept (contd)

Two-port device

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Full uncertainty matrix: one (8 x 8) matrix

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## Concept (contd)

### Two-port device

The off-diagonal elements (**shown in orange**) are used to represent the interactions between a component of one $S$-parameter and a component of a different $S$-parameter.

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Concept (contd)

Two-port device

The off-diagonal elements (shown in orange) are used to represent the interactions between a component of one S-parameter and a component of a different S-parameter.

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Concept (contd)

Without loss of generality, we examine just $S_{11}$ and $S_{21}$

Full uncertainty matrix: one (4 x 4) matrix

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Concept (contd)

First, we have the (2 x 2) sub-matrix for $S_{11}$: $(S_{11R}, S_{11I})$

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\[
\begin{pmatrix}
  u^2(S_{11R}) & u(S_{11R}, S_{11I}) \\
  u(S_{11I}, S_{11R}) & u^2(S_{11I})
\end{pmatrix}
\]

shows the interaction between $S_{11R}$ and $S_{11I}$
and the (2 x 2) sub-matrix for $S_{21}$: $(S_{21R}, S_{21I})$

\[
\begin{pmatrix}
  u^2(S_{21R}) & u(S_{21R}, S_{21I}) \\
  u(S_{21I}, S_{21R}) & u^2(S_{21I})
\end{pmatrix}
\]

shows the interaction between $S_{21R}$ and $S_{21I}$
Concept (contd)

But we can also have the (2 x 2) sub-matrix: \((S_{11_R}, S_{21_R})\)

<table>
<thead>
<tr>
<th></th>
<th>S_{11}</th>
<th>S_{21}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real</td>
<td>Imag</td>
</tr>
<tr>
<td>S_{11}</td>
<td>Real</td>
<td>(u^2(S_{11_R}))</td>
</tr>
<tr>
<td></td>
<td>Imag</td>
<td>(u(S_{11_I}, S_{11_R}))</td>
</tr>
<tr>
<td>S_{21}</td>
<td>Real</td>
<td>(u(S_{21_R}, S_{11_R}))</td>
</tr>
<tr>
<td></td>
<td>Imag</td>
<td>(u(S_{21_I}, S_{11_R}))</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
    u^2(S_{11_R}) & u(S_{11_R}, S_{21_R}) \\
    u(S_{21_R}, S_{11_R}) & u^2(S_{21_R})
\end{pmatrix}
\]

shows the interaction between \(S_{11_R}\) and \(S_{21_R}\)
Concept (contd)

And also the (2 x 2) sub-matrix: \((S_{11R}, S_{21I})\)

\[
\begin{pmatrix}
 u^2(S_{11R}) & u(S_{11R}, S_{21I}) \\
 u(S_{21I}, S_{11R}) & u^2(S_{21I})
\end{pmatrix}
\]

shows the interaction between \(S_{11R}\) and \(S_{21I}\)
Concept (contd)

and the (2 x 2) sub-matrix: \((S_{11I}, S_{21R})\)

<table>
<thead>
<tr>
<th></th>
<th>(S_{11I})</th>
<th>(S_{21R})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_{11I})</td>
<td>(u^2(S_{11R}))</td>
<td>(u(S_{11R}, S_{11I}))</td>
</tr>
<tr>
<td>(S_{21R})</td>
<td>(u(S_{11I}, S_{11R}))</td>
<td>(u^2(S_{11I}))</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
  u^2(S_{11I}) & u(S_{11I}, S_{21R}) \\
  u(S_{21R}, S_{11I}) & u^2(S_{21R})
\end{pmatrix}
\]

shows the interaction between \(S_{11I}\) and \(S_{21R}\)
And finally the (2 x 2) sub-matrix: \((S_{11I}, S_{21I})\)

\[
\begin{pmatrix}
S_{11I} & S_{21I} \\
S_{21I} & S_{11I}
\end{pmatrix}
\]

shows the interaction between \(S_{11I}\) and \(S_{21I}\)
Concept (contd)

So, the full (4 x 4) uncertainty matrix can effectively be represented using 6 two-dimensional vector sub-measurands

- \((S_{11R}, S_{11I})\) \(S_{11}\)
- \((S_{21R}, S_{21I})\) \(S_{21}\)
- \((S_{11R}, S_{21R})\)
- \((S_{11R}, S_{21I})\)
- \((S_{11I}, S_{21R})\)
- \((S_{11I}, S_{21I})\)
Example – measurements

- Mismatch two-port – designed and built by OML, Inc
- WR-10 waveguide
- 75 GHz to 110 GHz
- Measure: \( S_{11} \) and \( S_{21} \)

\[
S_{11R} \quad S_{11I} \quad S_{21R} \quad S_{21I}
\]
Beatty line - measurements

Magnitude of $S_{11}$ and $S_{21}$, derived from the S-parameter measurements
Beatty line – measurements (contd)

Magnitude of the vectors \( (S_{11R}, S_{21R}) \) and \( (S_{11I}, S_{21I}) \), derived from the S-parameter measurements
Beatty line – measurements (contd)

Magnitude of the vectors \((S_{11R}, S_{21I})\) and \((S_{11I}, S_{21R})\), derived from the S-parameter measurements
Uncertainty Ellipses

(2 x 2) uncertainty matrices

$$V = \begin{pmatrix} s^2(\bar{x}) & s(\bar{x}, \bar{y}) \\ s(\bar{y}, \bar{x}) & s^2(\bar{y}) \end{pmatrix}$$

can be used to construct ellipses to show regions of uncertainty

$$(S - \bar{S})^T V^{-1} (S - \bar{S}) = k^2$$

where $k$ is a coverage factor
Uncertainty Ellipses (contd)

Uncorrelated uncertainties . . .

Positively correlated uncertainties . . .

Negatively correlated uncertainties . . .
Uncertainty ellipses for 6 two-dimensional vector sub-measurands

- \((S_{11,R}, S_{11,I})\) for \(S_{11}\)
- \((S_{21,R}, S_{21,I})\) for \(S_{21}\)
- \((S_{11,R}, S_{21,R})\)
- \((S_{11,R}, S_{21,I})\)
- \((S_{11,I}, S_{21,R})\)
- \((S_{11,I}, S_{21,I})\)
Uncertainty ellipses for 6 two-dimensional vector sub-measurands

- \((S_{11R}, S_{11I})\) \(S_{11}\)
- \((S_{21R}, S_{21I})\) \(S_{21}\)
- \((S_{11R}, S_{21R})\)
- \((S_{11R}, S_{21I})\)
- \((S_{11I}, S_{21R})\)
- \((S_{11I}, S_{21I})\)
Uncertainty ellipses for 6 two-dimensional vector sub-measurands

- \( (S_{11R}, S_{11I}) \)
- \( (S_{21R}, S_{21I}) \)
- \( (S_{11R}, S_{21R}) \)
- \( (S_{11R}, S_{21I}) \)
- \( (S_{11I}, S_{21R}) \)
- \( (S_{11I}, S_{21I}) \)
Summary

- Multiple S-parameter measurements require a fully populated uncertainty matrix to capture all necessary information.
  - For example, for a two-port device, the four S-parameters require a (8 x 8) uncertainty matrix.
- The diagonal elements give the uncertainties in the components of each S-parameter.
- The off-diagonal elements characterize the interactions between components of different S-parameters.
Summary (contd)

- Multiple (2 x 2) sub-matrices can be used to evaluate and explore structure in the full uncertainty matrix.
- These representations can be useful for propagation of uncertainties and for comparing different sets of measurements (to demonstrate equivalence).
- To do this, multiple S-parameters should be considered as a single (higher dimensional) vector measurand.
  - For example, for a two-port device, the four S-parameters can be represented using a single 8-dimensional vector measurand, with an associated (8 x 8) uncertainty matrix.
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The authors thank Yuenie Lau (OML, Inc) for the loan of the mismatch waveguide line.

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