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Using Uncertainty Matrices to Analyse Measurements of Multiple *S*-parameters

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National
Measurement
System

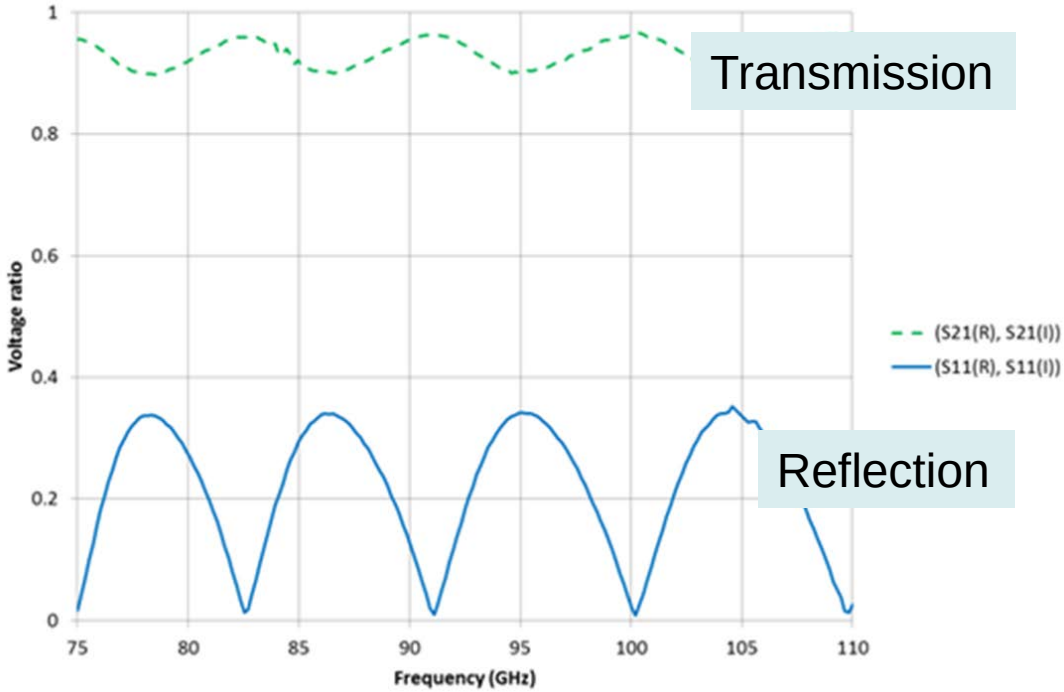
Overview

- Introduction
- Concept
- Example
- Uncertainty Ellipses
- Discussion
- Conclusion

Introduction

- Microwave measurements often contain interactions between multiple measurement parameters

- For example, for a mismatch line (Beatty line), as input reflection goes up, forward transmission goes down



Introduction (contd)

In this talk:

- Use uncertainty matrices to represent the interactions between measurement parameters
- Multiple (>1) complex-valued S-parameters

Why uncertainty matrices?

- Because these are useful when propagating uncertainties from one set of parameters to other parameters

Introduction (contd)

- For simplicity (here):
  - Concentrate on ‘just’ a 2-port device
    - Later on, we will assume symmetry ($S_{11} = S_{22}$)
    - And we will assume reciprocity ($S_{21} = S_{12}$)
- The approach generalizes to *n*-port devices that are non-symmetric and non-reciprocal

Concept

One-port device

One complex-valued S-parameter, $S_{11} \equiv \left(S_{11_{\mathrm{R}}}, S_{11_{\mathrm{I}}}\right)$

Uncertainty matrix:

$$\begin{pmatrix} u^2(S_{11_{\mathrm{R}}}) & u(S_{11_{\mathrm{R}}}, S_{11_{\mathrm{I}}}) \\ u\left(S_{11_{\mathrm{I}}}, S_{11_{\mathrm{R}}}\right) & u^2(S_{11_{\mathrm{I}}}) \end{pmatrix}$$

Concept (contd)

One-port device

The (2 x 2) uncertainty matrix shows the **uncertainty in each component** of the complex-valued S-parameter

i.e. $S_{11_{\mathrm{R}}}$ and $S_{11_{\mathrm{I}}}$

$$\begin{pmatrix} u^2(S_{11_{\mathrm{R}}}) & u(S_{11_{\mathrm{R}}}, S_{11_{\mathrm{I}}}) \\ u(S_{11_{\mathrm{I}}}, S_{11_{\mathrm{R}}}) & u^2(S_{11_{\mathrm{I}}}) \end{pmatrix}$$

Concept (contd)

One-port device

The (2 x 2) uncertainty matrix also shows the **interaction between each component** of the complex-valued S-parameter

i.e. between $S_{11_{\mathrm{R}}}$ and $S_{11_{\mathrm{I}}}$

$$\begin{pmatrix} u^2(S_{11_{\mathrm{R}}}) & u(S_{11_{\mathrm{R}}}, S_{11_{\mathrm{I}}}) \\ u(S_{11_{\mathrm{I}}}, S_{11_{\mathrm{R}}}) & u^2(S_{11_{\mathrm{I}}}) \end{pmatrix}$$

Concept (contd)

Two-port device

Four complex-valued S-parameters, S_{11}, S_{21}, S_{12}, S_{22}

Four (2 x 2) uncertainty matrices: one for each S-parameter

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\ u(S_{11_I}, S_{11_R}) & u^2(S_{11_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{21_R}) & u(S_{21_R}, S_{21_I}) \\ u(S_{21_I}, S_{21_R}) & u^2(S_{21_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{12_R}) & u(S_{12_R}, S_{12_I}) \\ u(S_{12_I}, S_{12_R}) & u^2(S_{12_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{22_R}) & u(S_{22_R}, S_{22_I}) \\ u(S_{22_I}, S_{22_R}) & u^2(S_{22_I}) \end{pmatrix}$$

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Concept (contd)

Two-port device

Each (2 x 2) uncertainty matrix shows the **uncertainty in, and interaction between**, each component in each complex-valued S-parameter, S_{11}, S_{21}, S_{12}, S_{22}

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\ u(S_{11_I}, S_{11_R}) & u^2(S_{11_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{21_R}) & u(S_{21_R}, S_{21_I}) \\ u(S_{21_I}, S_{21_R}) & u^2(S_{21_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{12_R}) & u(S_{12_R}, S_{12_I}) \\ u(S_{12_I}, S_{12_R}) & u^2(S_{12_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{22_R}) & u(S_{22_R}, S_{22_I}) \\ u(S_{22_I}, S_{22_R}) & u^2(S_{22_I}) \end{pmatrix}$$

Concept (contd)

Two-port device

Each (2 x 2) uncertainty matrix shows the **uncertainty in, and interaction between**, each component in each complex-valued S-parameter, S_{11}, S_{21}, S_{12}, S_{22}

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\ u(S_{11_I}, S_{11_R}) & u^2(S_{11_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{21_R}) & u(S_{21_R}, S_{21_I}) \\ u(S_{21_I}, S_{21_R}) & u^2(S_{21_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{12_R}) & u(S_{12_R}, S_{12_I}) \\ u(S_{12_I}, S_{12_R}) & u^2(S_{12_I}) \end{pmatrix}$$

$$\begin{pmatrix} u^2(S_{22_R}) & u(S_{22_R}, S_{22_I}) \\ u(S_{22_I}, S_{22_R}) & u^2(S_{22_I}) \end{pmatrix}$$

Concept (contd)

Two-port device

But these (2 x 2) uncertainty matrices **do not** show the **interaction between** a **component of one S-parameter** and a **component of a different S-parameter**

For example, for the interaction between $S_{11_{\mathrm{R}}}$ and $S_{21_{\mathrm{R}}}$, we need the uncertainty matrix:

$$\begin{pmatrix} u^2(S_{11_{\mathrm{R}}}) & u(S_{11_{\mathrm{R}}}, S_{21_{\mathrm{R}}}) \\ u(S_{21_{\mathrm{R}}}, S_{11_{\mathrm{R}}}) & u^2(S_{21_{\mathrm{R}}}) \end{pmatrix}$$

Concept (contd)

Two-port device

Or, for the interaction between $S_{11_{\mathrm{R}}}$ and $S_{21_{\mathrm{I}}}$,

we need the uncertainty matrix:

$$\begin{pmatrix} u^2(S_{11_{\mathrm{R}}}) & u(S_{11_{\mathrm{R}}}, S_{21_{\mathrm{I}}}) \\ u(S_{21_{\mathrm{I}}}, S_{11_{\mathrm{R}}}) & u^2(S_{21_{\mathrm{I}}}) \end{pmatrix}$$

Concept (contd)

Two-port device

To represent the interaction between:

- **both** components (real and imaginary)
- of **all four** S-parameters (S_{11}, S_{21}, S_{12}, S_{22})

We need a single ([4x2] x [4x2]) matrix

≡ (8 x 8) matrix

Concept (contd)

Two-port device

Four complex-valued S-parameters, S_{11}, S_{21}, S_{12}, S_{22}

Full uncertainty matrix: one (8 x 8) matrix

		S_{11}		S_{21}		S_{12}		S_{22}	
		Real	Imag	Real	Imag	Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$	$u(S_{11_R}, S_{12_R})$	$u(S_{11_R}, S_{12_I})$	$u(S_{11_R}, S_{22_R})$	$u(S_{11_R}, S_{22_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$	$u(S_{11_I}, S_{12_R})$	$u(S_{11_I}, S_{12_I})$	$u(S_{11_I}, S_{22_R})$	$u(S_{11_I}, S_{22_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$	$u(S_{21_R}, S_{12_R})$	$u(S_{21_R}, S_{12_I})$	$u(S_{21_R}, S_{22_R})$	$u(S_{21_R}, S_{22_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$	$u(S_{21_I}, S_{12_R})$	$u(S_{21_I}, S_{12_I})$	$u(S_{21_I}, S_{22_R})$	$u(S_{21_I}, S_{22_I})$
S_{12}	Real	$u(S_{12_R}, S_{11_R})$	$u(S_{12_R}, S_{11_I})$	$u(S_{12_R}, S_{21_R})$	$u(S_{12_R}, S_{21_I})$	$u^2(S_{12_R})$	$u(S_{12_R}, S_{12_I})$	$u(S_{12_R}, S_{22_R})$	$u(S_{12_R}, S_{22_I})$
	Imag	$u(S_{12_I}, S_{11_R})$	$u(S_{12_I}, S_{11_I})$	$u(S_{12_I}, S_{21_R})$	$u(S_{12_I}, S_{21_I})$	$u(S_{12_I}, S_{12_R})$	$u^2(S_{12_I})$	$u(S_{12_I}, S_{22_R})$	$u(S_{12_I}, S_{22_I})$
S_{22}	Real	$u(S_{22_R}, S_{11_R})$	$u(S_{22_R}, S_{11_I})$	$u(S_{22_R}, S_{21_R})$	$u(S_{22_R}, S_{21_I})$	$u(S_{22_R}, S_{12_R})$	$u(S_{22_R}, S_{12_I})$	$u^2(S_{22_R})$	$u(S_{22_R}, S_{22_I})$
	Imag	$u(S_{22_I}, S_{11_R})$	$u(S_{22_I}, S_{11_I})$	$u(S_{22_I}, S_{21_R})$	$u(S_{22_I}, S_{21_I})$	$u(S_{22_I}, S_{12_R})$	$u(S_{22_I}, S_{12_I})$	$u(S_{22_I}, S_{22_R})$	$u^2(S_{22_I})$

Concept (contd)

Two-port device

Four complex-valued S-parameters, S_{11}, S_{21}, S_{12}, S_{22}

Full uncertainty matrix: one (8 x 8) matrix

		S_{11}		S_{21}		S_{12}		S_{22}	
		Real	Imag	Real	Imag	Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$	$u(S_{11_R}, S_{12_R})$	$u(S_{11_R}, S_{12_I})$	$u(S_{11_R}, S_{22_R})$	$u(S_{11_R}, S_{22_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$	$u(S_{11_I}, S_{12_R})$	$u(S_{11_I}, S_{12_I})$	$u(S_{11_I}, S_{22_R})$	$u(S_{11_I}, S_{22_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$	$u(S_{21_R}, S_{12_R})$	$u(S_{21_R}, S_{12_I})$	$u(S_{21_R}, S_{22_R})$	$u(S_{21_R}, S_{22_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$	$u(S_{21_I}, S_{12_R})$	$u(S_{21_I}, S_{12_I})$	$u(S_{21_I}, S_{22_R})$	$u(S_{21_I}, S_{22_I})$
S_{12}	Real	$u(S_{12_R}, S_{11_R})$	$u(S_{12_R}, S_{11_I})$	$u(S_{12_R}, S_{21_R})$	$u(S_{12_R}, S_{21_I})$	$u^2(S_{12_R})$	$u(S_{12_R}, S_{12_I})$	$u(S_{12_R}, S_{22_R})$	$u(S_{12_R}, S_{22_I})$
	Imag	$u(S_{12_I}, S_{11_R})$	$u(S_{12_I}, S_{11_I})$	$u(S_{12_I}, S_{21_R})$	$u(S_{12_I}, S_{21_I})$	$u(S_{12_I}, S_{12_R})$	$u^2(S_{12_I})$	$u(S_{12_I}, S_{22_R})$	$u(S_{12_I}, S_{22_I})$
S_{22}	Real	$u(S_{22_R}, S_{11_R})$	$u(S_{22_R}, S_{11_I})$	$u(S_{22_R}, S_{21_R})$	$u(S_{22_R}, S_{21_I})$	$u(S_{22_R}, S_{12_R})$	$u(S_{22_R}, S_{12_I})$	$u^2(S_{22_R})$	$u(S_{22_R}, S_{22_I})$
	Imag	$u(S_{22_I}, S_{11_R})$	$u(S_{22_I}, S_{11_I})$	$u(S_{22_I}, S_{21_R})$	$u(S_{22_I}, S_{21_I})$	$u(S_{22_I}, S_{12_R})$	$u(S_{22_I}, S_{12_I})$	$u(S_{22_I}, S_{22_R})$	$u^2(S_{22_I})$

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Concept (contd)

Two-port device

The off-diagonal elements (**shown in orange**) are used to represent the interactions between a component of one *S*-parameter and a component of a different *S*-parameter

		S_{11}		S_{21}		S_{12}		S_{22}	
		Real	Imag	Real	Imag	Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R},S_{11_I})$	$u(S_{11_R},S_{21_R})$	$u(S_{11_R},S_{21_I})$	$u(S_{11_R},S_{12_R})$	$u(S_{11_R},S_{12_I})$	$u(S_{11_R},S_{22_R})$	$u(S_{11_R},S_{22_I})$
	Imag	$u(S_{11_I},S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I},S_{21_R})$	$u(S_{11_I},S_{21_I})$	$u(S_{11_I},S_{12_R})$	$u(S_{11_I},S_{12_I})$	$u(S_{11_I},S_{22_R})$	$u(S_{11_I},S_{22_I})$
S_{21}	Real	$u(S_{21_R},S_{11_R})$	$u(S_{21_R},S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R},S_{21_I})$	$u(S_{21_R},S_{12_R})$	$u(S_{21_R},S_{12_I})$	$u(S_{21_R},S_{22_R})$	$u(S_{21_R},S_{22_I})$
	Imag	$u(S_{21_I},S_{11_R})$	$u(S_{21_I},S_{11_I})$	$u(S_{21_I},S_{21_R})$	$u^2(S_{21_I})$	$u(S_{21_I},S_{12_R})$	$u(S_{21_I},S_{12_I})$	$u(S_{21_I},S_{22_R})$	$u(S_{21_I},S_{22_I})$
S_{12}	Real	$u(S_{12_R},S_{11_R})$	$u(S_{12_R},S_{11_I})$	$u(S_{12_R},S_{21_R})$	$u(S_{12_R},S_{21_I})$	$u^2(S_{12_R})$	$u(S_{12_R},S_{12_I})$	$u(S_{12_R},S_{22_R})$	$u(S_{12_R},S_{22_I})$
	Imag	$u(S_{12_I},S_{11_R})$	$u(S_{12_I},S_{11_I})$	$u(S_{12_I},S_{21_R})$	$u(S_{12_I},S_{21_I})$	$u(S_{12_I},S_{12_R})$	$u^2(S_{12_I})$	$u(S_{12_I},S_{22_R})$	$u(S_{12_I},S_{22_I})$
S_{22}	Real	$u(S_{22_R},S_{11_R})$	$u(S_{22_R},S_{11_I})$	$u(S_{22_R},S_{21_R})$	$u(S_{22_R},S_{21_I})$	$u(S_{22_R},S_{12_R})$	$u(S_{22_R},S_{12_I})$	$u^2(S_{22_R})$	$u(S_{22_R},S_{22_I})$
	Imag	$u(S_{22_I},S_{11_R})$	$u(S_{22_I},S_{11_I})$	$u(S_{22_I},S_{21_R})$	$u(S_{22_I},S_{21_I})$	$u(S_{22_I},S_{12_R})$	$u(S_{22_I},S_{12_I})$	$u(S_{22_I},S_{22_R})$	$u^2(S_{22_I})$

National Measurement System

Concept (contd)

Two-port device

The off-diagonal elements (**shown in orange**) are used to represent the interactions between a component of one *S*-parameter and a component of a different *S*-parameter

		S_{11}		S_{21}		S_{12}		S_{22}	
		Real	Imag	Real	Imag	Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R},S_{11_I})$	$u(S_{11_R},S_{21_R})$	$u(S_{11_R},S_{21_I})$	$u(S_{11_R},S_{12_R})$	$u(S_{11_R},S_{12_I})$	$u(S_{11_R},S_{22_R})$	$u(S_{11_R},S_{22_I})$
	Imag	$u(S_{11_I},S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I},S_{21_R})$	$u(S_{11_I},S_{21_I})$	$u(S_{11_I},S_{12_R})$	$u(S_{11_I},S_{12_I})$	$u(S_{11_I},S_{22_R})$	$u(S_{11_I},S_{22_I})$
S_{21}	Real	$u(S_{21_R},S_{11_R})$	$u(S_{21_R},S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R},S_{21_I})$	$u(S_{21_R},S_{12_R})$	$u(S_{21_R},S_{12_I})$	$u(S_{21_R},S_{22_R})$	$u(S_{21_R},S_{22_I})$
	Imag	$u(S_{21_I},S_{11_R})$	$u(S_{21_I},S_{11_I})$	$u(S_{21_I},S_{21_R})$	$u^2(S_{21_I})$	$u(S_{21_I},S_{12_R})$	$u(S_{21_I},S_{12_I})$	$u(S_{21_I},S_{22_R})$	$u(S_{21_I},S_{22_I})$
S_{12}	Real	$u(S_{12_R},S_{11_R})$	$u(S_{12_R},S_{11_I})$	$u(S_{12_R},S_{21_R})$	$u(S_{12_R},S_{21_I})$	$u^2(S_{12_R})$	$u(S_{12_R},S_{12_I})$	$u(S_{12_R},S_{22_R})$	$u(S_{12_R},S_{22_I})$
	Imag	$u(S_{12_I},S_{11_R})$	$u(S_{12_I},S_{11_I})$	$u(S_{12_I},S_{21_R})$	$u(S_{12_I},S_{21_I})$	$u(S_{12_I},S_{12_R})$	$u^2(S_{12_I})$	$u(S_{12_I},S_{22_R})$	$u(S_{12_I},S_{22_I})$
S_{22}	Real	$u(S_{22_R},S_{11_R})$	$u(S_{22_R},S_{11_I})$	$u(S_{22_R},S_{21_R})$	$u(S_{22_R},S_{21_I})$	$u(S_{22_R},S_{12_R})$	$u(S_{22_R},S_{12_I})$	$u^2(S_{22_R})$	$u(S_{22_R},S_{22_I})$
	Imag	$u(S_{22_I},S_{11_R})$	$u(S_{22_I},S_{11_I})$	$u(S_{22_I},S_{21_R})$	$u(S_{22_I},S_{21_I})$	$u(S_{22_I},S_{12_R})$	$u(S_{22_I},S_{12_I})$	$u(S_{22_I},S_{22_R})$	$u^2(S_{22_I})$

Concept (contd)

Without loss of generality, we examine just S_{11} and S_{21}

Full uncertainty matrix: one (4 x 4) matrix

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

Concept (contd)

First, we have the (2 x 2) sub-matrix for S_{11}: (S_{11_R}, S_{11_I})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{11_I}) \\ u(S_{11_I}, S_{11_R}) & u^2(S_{11_I}) \end{pmatrix}$$

shows the interaction between S_{11_R} and S_{11_I}

Concept (contd)

and the (2 x 2) sub-matrix for S_{21}: (S_{21_R}, S_{21_I})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{21_R}) & u(S_{21_R}, S_{21_I}) \\ u(S_{21_I}, S_{21_R}) & u^2(S_{21_I}) \end{pmatrix}$$

shows the interaction between S_{21_R} and S_{21_I}

Concept (contd)

But we can also have the (2 x 2) sub-matrix: (S_{11_R}, S_{21_R})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{21_R}) \\ u(S_{21_R}, S_{11_R}) & u^2(S_{21_R}) \end{pmatrix}$$

shows the interaction between S_{11_R} and S_{21_R}

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Concept (contd)

And also the (2 x 2) sub-matrix: (S_{11_R}, S_{21_I})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{11_R}) & u(S_{11_R}, S_{21_I}) \\ u(S_{21_I}, S_{11_R}) & u^2(S_{21_I}) \end{pmatrix}$$

shows the interaction between S_{11_R} and S_{21_I}

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Concept (contd)

and the (2 x 2) sub-matrix: (S_{11_I}, S_{21_R})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{11_I}) & u(S_{11_I}, S_{21_R}) \\ u(S_{21_R}, S_{11_I}) & u^2(S_{21_R}) \end{pmatrix}$$

shows the interaction between S_{11_I} and S_{21_R}

Concept (contd)

And finally the (2 x 2) sub-matrix: (S_{11_I}, S_{21_I})

		S_{11}		S_{21}	
		Real	Imag	Real	Imag
S_{11}	Real	$u^2(S_{11_R})$	$u(S_{11_R}, S_{11_I})$	$u(S_{11_R}, S_{21_R})$	$u(S_{11_R}, S_{21_I})$
	Imag	$u(S_{11_I}, S_{11_R})$	$u^2(S_{11_I})$	$u(S_{11_I}, S_{21_R})$	$u(S_{11_I}, S_{21_I})$
S_{21}	Real	$u(S_{21_R}, S_{11_R})$	$u(S_{21_R}, S_{11_I})$	$u^2(S_{21_R})$	$u(S_{21_R}, S_{21_I})$
	Imag	$u(S_{21_I}, S_{11_R})$	$u(S_{21_I}, S_{11_I})$	$u(S_{21_I}, S_{21_R})$	$u^2(S_{21_I})$

$$\begin{pmatrix} u^2(S_{11_I}) & u(S_{11_I}, S_{21_I}) \\ u(S_{11_I}, S_{21_I}) & u^2(S_{21_I}) \end{pmatrix}$$

shows the interaction between S_{11_I} and S_{21_I}

Concept (contd)

So, the full (4 x 4) uncertainty matrix can effectively be represented using 6 two-dimensional vector sub-measurands

- (S_{11_R}, S_{11_I}) $\quad S_{11}$
- (S_{21_R}, S_{21_I}) $\quad S_{21}$
- (S_{11_R}, S_{21_R})
- (S_{11_R}, S_{21_I})
- (S_{11_I}, S_{21_R})
- (S_{11_I}, S_{21_I})

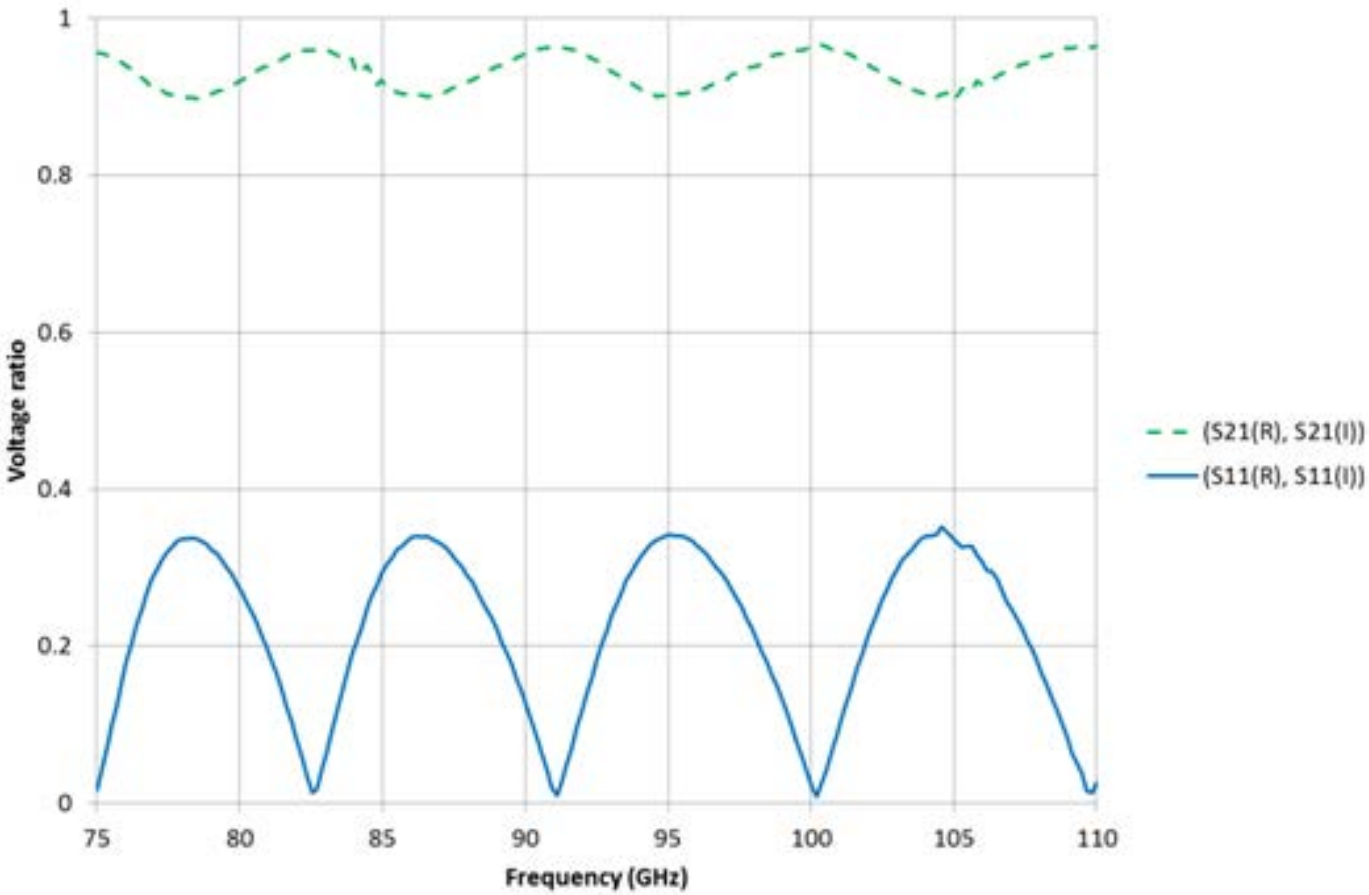
Example – measurements

- Mismatch two-port – designed and built by OML, Inc
- WR-10 waveguide
- 75 GHz to 110 GHz
- Measure: S_{11} and S_{21}

$$S_{11_{\mathrm{R}}} \quad S_{11_{\mathrm{I}}} \quad S_{21_{\mathrm{R}}} \quad S_{21_{\mathrm{I}}}$$

Beatty line - measurements

Magnitude of S_{11} and S_{21}, derived from the S-parameter measurements



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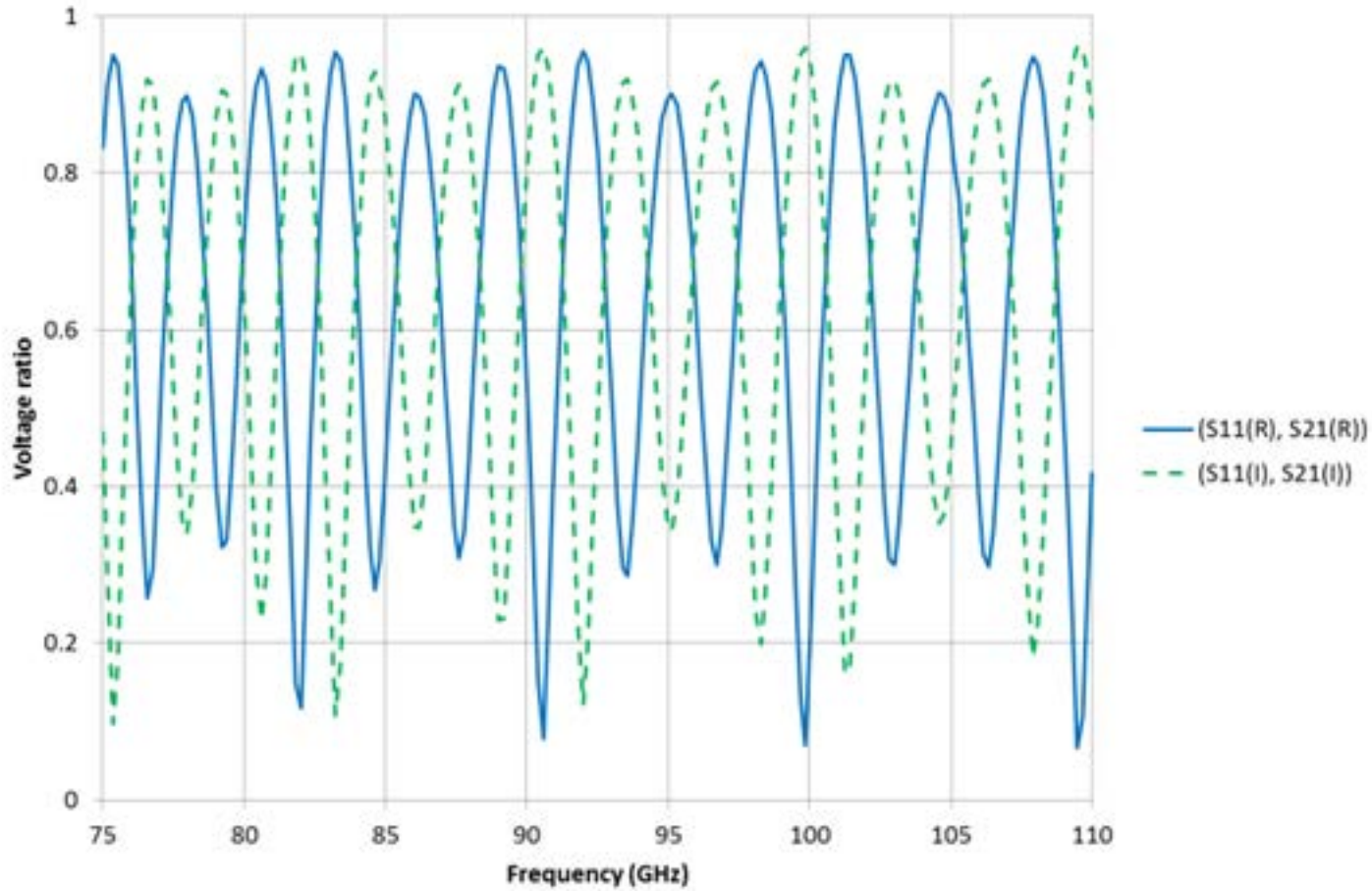
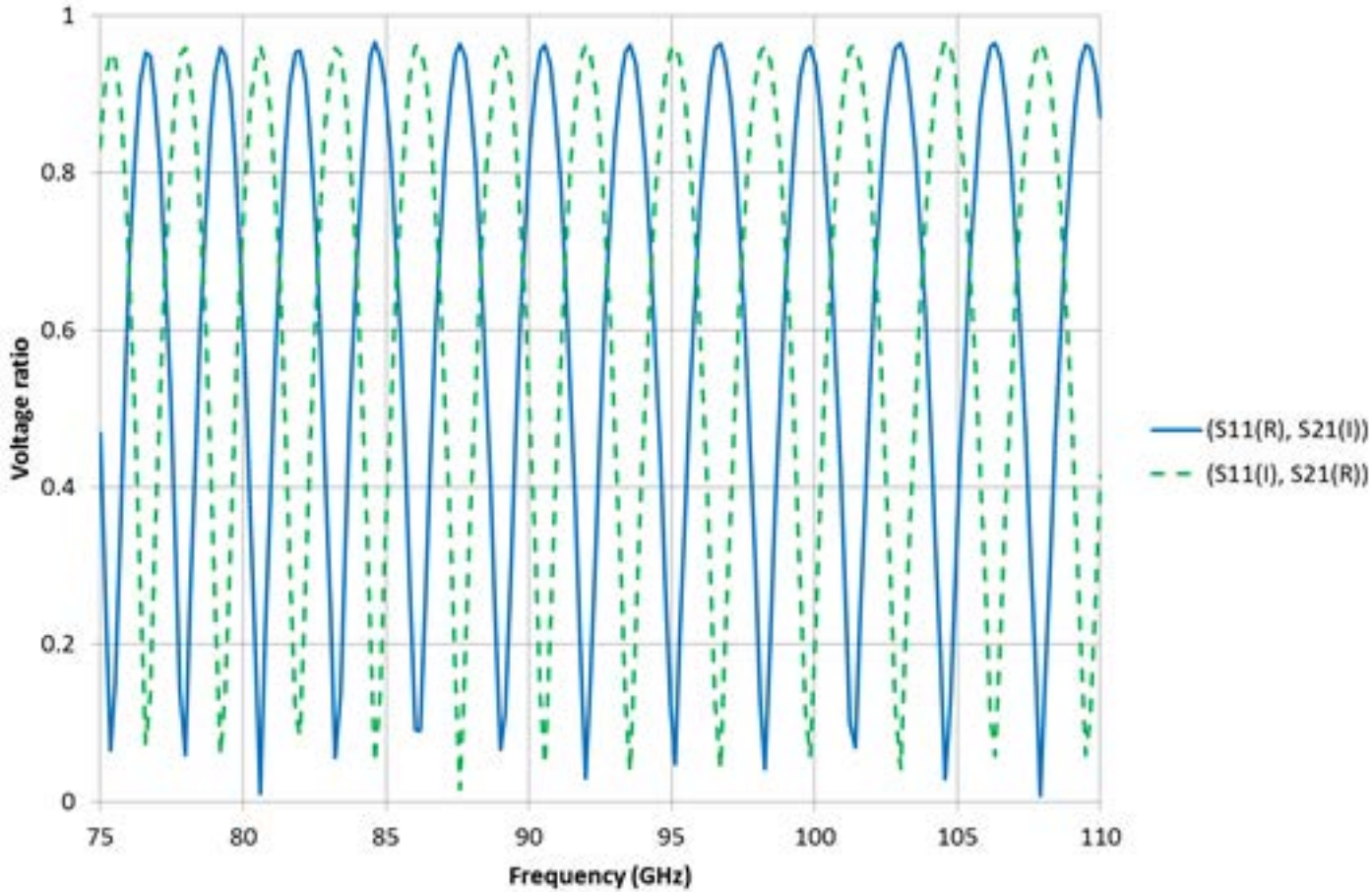
Beatty line – measurements (contd)

Magnitude of the vectors (S_{11_R}, S_{21_R}) and (S_{11_I}, S_{21_I}), derived from the *S*-parameter measurements



Beatty line – measurements (contd)

Magnitude of the vectors (S_{11_R}, S_{21_I}) and (S_{11_I}, S_{21_R}), derived from the *S*-parameter measurements



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Uncertainty Ellipses

(2 x 2) uncertainty matrices

$$V = \begin{pmatrix} s^2(\bar{x}) & s(\bar{x},\bar{y}) \\ s(\bar{y},\bar{x}) & s^2(\bar{y}) \end{pmatrix}$$

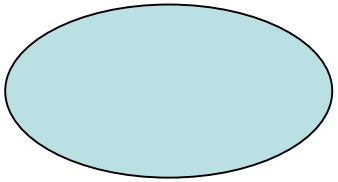
can be used to construct ellipses to show regions of uncertainty

$$(S - \bar{S})^{\mathrm{T}} V^{-1} (S - \bar{S}) = k^2$$

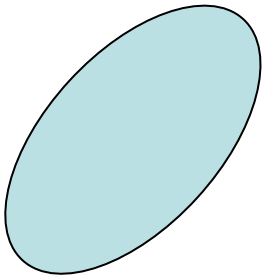
where *k* is a coverage factor

Uncertainty Ellipses (contd)

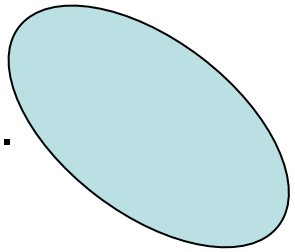
Uncorrelated uncertainties . . .



Positively correlated uncertainties . . .



Negatively correlated uncertainties . . .



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Uncertainty ellipses
for 6 two-dimensional vector sub-measurands

- $(\boldsymbol{S}_{11_R}, \boldsymbol{S}_{11_I})$ $\boldsymbol{S}_{11}$
- $(\boldsymbol{S}_{21_R}, \boldsymbol{S}_{21_I})$ $\boldsymbol{S}_{21}$
- $(S_{11_\mathrm{R}}, S_{21_\mathrm{R}})$
- $(S_{11_\mathrm{R}}, S_{21_\mathrm{I}})$
- $(S_{11_\mathrm{I}}, S_{21_\mathrm{R}})$
- $(S_{11_\mathrm{I}}, S_{21_\mathrm{I}})$

Animation 1

Uncertainty ellipses
for 6 two-dimensional vector sub-measurands

- (S_{11_R}, S_{11_I}) S_{11}
- (S_{21_R}, S_{21_I}) S_{21}
- $\boldsymbol{(S_{11_R}, S_{21_R})}$
- $\boldsymbol{(S_{11_R}, S_{21_I})}$
- (S_{11_I}, S_{21_R})
- (S_{11_I}, S_{21_I})

Animation 2

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Uncertainty ellipses for 6 two-dimensional vector sub-measurands

- (S_{11_R}, S_{11_I}) S_{11}
- (S_{21_R}, S_{21_I}) S_{21}
- (S_{11_R}, S_{21_R})
- (S_{11_R}, S_{21_I})
- $\boldsymbol{(S_{11_I}, S_{21_R})}$
- $\boldsymbol{(S_{11_I}, S_{21_I})}$

Animation 3

Summary

- Multiple *S*-parameter measurements require a fully populated uncertainty matrix to capture all necessary information
  - For example, for a two-port device, the four *S*-parameters require a (8 x 8) uncertainty matrix
- The diagonal elements give the uncertainties in the components of each *S*-parameter
- The off-diagonal elements characterize the interactions between components of different *S*-parameters

Summary (contd)

- Multiple (2 x 2) sub-matrices can be used to evaluate and explore structure in the full uncertainty matrix
- These representations can be useful for propagation of uncertainties and for comparing different sets of measurements (to demonstrate equivalence)
- To do this, multiple *S*-parameters should be considered as a single (higher dimensional) vector measurand
  - For example, for a two-port device, the four *S*-parameters can be represented using a single 8-dimensional vector measurand, with an associated (8 x 8) uncertainty matrix

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